ADVANCED LEVEL NATIONAL EXAMINATIONS 2011

SUBJECT: MATHEMATICS II

COMBINATIONS:  - MATHS-CHEMISTRY-BIOLOGY: MCB
                 - MATHS-COMPUTER SCIENCE-ECONOMICS: MCE
                 - MATHS-ECONOMICS-GEOGRAPHY: MEG
                 - MATHS-PHYSICS-COMPUTER SCIENCE: MPC
                 - MATHS-PHYSICS-GEOGRAPHY: MPG
                 - PHYSICS-CHEMISTRY-MATHS: PCM
                 - PHYSICS-ECONOMICS-MATHS: PEM

TIME: 3 HOURS

INSTRUCTIONS:

- This paper consists in two sections: A and (B)

Section A : Attempt all questions. (55 marks)
Section B: Attempt any three questions. (45 marks)

Geometrical instruments and silent non-programmable calculators may be used.
SECTION A: Attempt all questions. (55 marks)

1. What values (real numbers) of \( x \) satisfying the following condition:
   \( 4(x+5) - 6(2x+3) = 3(x+14) - 2(5-x)+9 \)  
   \( \text{as } \left| a \right| = \begin{cases} 
   a & \text{if } a \geq 0 \\
   -a & \text{if } a \leq 0
   \end{cases} \)  
   (2.5 marks)

2. If \((u,v,w)\) is a basis of the real vector space \( \mathbb{R}^3 \) determine whether or not \((u+v, u+2w, u-w)\) is also a basis of \( \mathbb{R}^3 \). (2 marks)

3. Let \( f: \mathbb{R} \to \mathbb{R} : f(x) = \frac{x^2 - 1}{|x-1|} \)  
   (a) Find \( \lim_{x \to 1^+} f(x) \) and \( \lim_{x \to 1^-} f(x) \). (3.5 marks)
   (b) Discuss the limit of \( f(x) \) as \( x \) approaches 1. (0.5 mark)
   (c) Sketch the graph of \( f(x) \). (1 mark)

4. Express \( f(x) = \frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6} \) in partial fractions. Then find antiderivative of \( f(x) \). (4.5 marks)

5. Find the number of ways that 6 teachers can be assigned to 4 sections of mathematics course if no teacher is assigned to more than one section. (2.5 marks)

6. In Euclidian space, find an equation for the plane consisting of all points that are equidistant from the points \((-4, 2, 1)\) and \((2, -4, 3)\). (3 marks)

7. Find any asymptotes of the function \( f \) if \( f(x) = \frac{x-1}{x^2-1} \). (3 marks)

8. Find a second degree polynomial \( P(x) \) such that \( P(2)=5 \), \( P'(2)=3 \) and \( P''(2)=2 \) where \( P' \) and \( P'' \) are first and second derivatives of \( P \) respectively. (4 marks)

9. (a) Evaluate the derivative of \( f(x) = \ln(x^3 + 7x^2) \) where \( \ln \) stands for natural logarithm function; (1.5 marks)
   (b) and evaluate the integral \( \int \frac{x^3}{x^4 + 7} dx \) (1.5 marks)

10. How many distinct permutations can be made from the letters of the word \( \text{"infinity"} \)? (2 marks)
11. Find the domain and the derivative of the numerical function \( f(x) = \frac{x}{1 - \ln(x - 1)} \). (4 marks)

12. The probability that a patient recovers from a delicate heart operation is 0.8. What is the probability that
(a) Exactly 2 of the next 3 patients who have this operation survive? (1.5 marks)
(b) All of the next 3 patients who have this operation survive? (1.5 marks)

13. Find the value of the complex number \( Z = \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right)^{2010} \).

Leaving your answer in standard form \( Z = a + bi \). (4 marks)

14. Sketch the plane region bounded by \( y = x \) and \( y = x^2 - 6 \).
Then estimate the volume generated by this region when revolved about \( x = -3 \). (5.5 marks)

15. The numbers of incorrect answers on a true-false competency test for a random sample of 15 students were recorded as follows: 2, 1, 3, 0, 1, 3, 6, 0, 3, 3, 5, 2, 1, 4, and 2.
Find the mean, the median, the mode and the sample standard deviation. (4 marks)

SECTION B: Attempt any three questions. (45 marks)

16. Let \( f: \mathbb{R} \to \mathbb{R} : f(x) = |x - 2| - 1 + \frac{1}{x^2} \).
(a) Determine the domain of \( f(x) \). (2 marks)
(b) Write \( f(x) \) without signs of absolute value. (2 marks)
(c) Study the derivability of \( f(x) \) at \( x = -2 \). (4 marks)
(d) Evaluate the limit of \( f(x) - (x + 1) \) when \( x \) approaches \( -\infty \)
and the limit of \( f(x) - (x - 3) \) when \( x \) approaches \( +\infty \).
Is there any relationship between lines \( y = x - 3 \)
and \( y = -x + 1 \), and the graph of the function \( f \)? (3 marks)
(e) Evaluate \( \int f(x) \, dx \). (4 marks)

17. (a) Solve the equation \( n^{-1}C_{n-5} = 3 \cdot n^{-3}C_{n-7} \) (or \( \left( \binom{n-1}{n-5} \right) = 3 \cdot \left( \binom{n-3}{n-7} \right) \))
in the set of positive integers. (6.5 marks)
(b) Consider the quadratic polynomial \( z^2 - 6z + c \) where \( c \) is real. For what values of \( c \) does this polynomial have real roots? (2.5 marks)

(c) Multiply out the expression \((z + 7)(z^2 - 6z + 25)\) and hence find all roots (real or complex) of the polynomial \( z^3 + z^2 - 17z + 175 \). (6 marks)

18. (a) In Euclidean space, find vector, parametric and symmetric equations for

(i) the line through origin and the point \((1,2,3)\) (4 marks)

(ii) The line through \((0,2,-1)\) and parallel to the line with parametric equations \[
\begin{align*}
x &= 1 + 2t \\
y &= 3t \\
z &= 5 - 7t
\end{align*}
\] (4 marks)

(b) Find all cube roots of the complex number \( W = -1 + i \). Leave your answer in polar form and trigonometric form. (7 marks)

19. (a) Compute the sixth degree Taylor polynomial generated by \( f(x) = \ln x \) about \( x = 1 \).

Using this result, evaluate \( \lim_{x \to 1} \frac{\ln x}{x - 1} \). (6 marks)

(b) Find all values of \( x \) that satisfy

(i) the inequality \( 2 \cos(x) + 1 \geq 0 \) in the interval \([0, 2\pi]\); (2.5 marks)

(ii) the equality \( \log_2 \left( \frac{x^2 - 1}{x + 1} \right) = 1 \) (3.5 marks)

(c) If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and one dictionary, what is the probability that

(i) The dictionary is selected? (1.5 marks)

(ii) 2 novels and 1 book of poems are selected? (1.5 marks)

20. (a) Let \( * \) be a binary operation defined on the set \( Z \) of all integers by \( x * y = x + y + 3 \).

Determine whether the operation is commutative, and whether there is an identity element.

Can you find a symmetric (inverse) of any integer? (8 marks)

(b) Sketch and estimate the area of the region bounded by the curves \( y = x \) and \( y = x^2 - 2 \). (7 marks)